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POINTWISE CONTINUOUS SHADOWING AND STABILITY IN GROUP ACTIONS

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Abstract. Let $Act(G, X)$ be the set of all continuous actions of a finitely generated group G on a compact metric space X. In this paper, we study the concepts of topologically stable points and continuous shadowable points of a group action $T \in Act(G, X)$. We show that if T is expansive then the set of continuous shadowable points is contained in the set of topologically stable points.

1. Introduction

The shadowing property was first established for systems generated by hyperbolic diffeomorphisms and later for those generated by hyperbolic homeomorphisms. It refers to the general problem of approximated by a true orbit in the presence of noise or round-off error. Lee [6] introduced and discussed the notions of continuous shadowing property so there are several way to define the shadowing property for homeomorphisms.

Pilyugin *et al.* [8, 11] introduced the notions of shadowing for finitely generated group actions which are generalizations of shadowing for homeomorphisms on compact metric spaces. Recently, Morales [7] generalized the definition of shadowing for homeomorphisms in a compact metric space by splitting the shadowing property into pointwise shadowings giving rise to the concept of shadowable points, which are points where the shadowing property holds for pseudo-orbits passing through them. Kawaguchi [3] also presented shadowable points and extended shadowable points to the concept of quantitative version for

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homeomorphisms. Afterward, Kim and Lee [4] introduced the notion of shadowable points, nonwandering set, chain recurrent set for finitely generated group actions.

Chung and Lee [1] introduced the notion of topological stability for finitely generated group actions. Very recently, by splitting the topological stability into pointwise stability, Koo *et al.* [5] introduced the notion of topologically stable point for homeomorphisms on compact metric spaces, which enables us to analyze the local aspect of the topological stability and proved pointwise version of Walters' stability theorem.

In this paper, we introduce the notion of continuous shadowable points and topologically stable points for a finitely generated group actions. Furthermore, we prove the following properties: the set of continuous shadowable points is an invariant Borel set and the set of topologically stable points is also invariant. Since the finitely generated group G is not necessarily abelian, we observe that these extensions are really weaker than ones of homeomorphisms. Finally, we show that the relations between the set of continuous shadowable points and the set of topologically stable points for an expansive action.

2. Preliminaries

First of all, we round out the introduction with some notations. Let G be a finitely generated group with the discrete topology and X be a compact metric space with a metric d. We denote $Homeo(X)$ by the space of all homeomorphisms of X and $Act(G, X)$ by the set of all continuous actions T of G on X; *i.e.* $T : G \times X \to X$ is a continuous map such that $T(g, \cdot)$ is continuous, $T(e, x) = x$ and $T(g, T(h, x)) = T(gh, x)$ for $x \in X$ and $g, h \in G$, where e is the identity element of G. For simplicity, $T(g, x)$ will be denoted by $T_g(x)$. Let $Homeo(X)^G = \prod_G Homeo(X)$ be the set of homeomorphisms from G to $Homeo(X)$ with the product topology. Then $Act(G, X)$ can be considered as a subset of $Homeo(X)^G$. Let A be a symmetric finitely generating set of G, i.e., for any $a \in A$, $a^{-1} \in A$. If A is a finintely generating set of G, then there always exists a symmetric finitely generating set containing A. Throughout the paper, a finitely generating set A of G means a symmetric finitely generating set. We define a metric d_A on $Act(G, X)$ by

$$d_A(T, S) = \sup\{d(T_a x, S_a x) : x \in X, a \in A\}$$

for $T, S \in Act(G, X)$. Then the topology on $Act(G, X)$ induced by d_A coincides with the product topology on $Act(G, X)$ inherited from

$Homeo(X)^G$. Hence the space $Act(G, X)$ is a separable complete metrizable topological space, and so a Baire space.

We say that a sequence of points $\{x_g\}_{g\in G} \subset X$ is a δ-pseudo orbit for T with respect to A if $d(T_a(x_g), x_{ag}) < \delta$ for every $a \in A,\ g \in G$. A δ-pseudo orbit $\{x_g\}_{g\in G}$ for T with respect to A is said to be ϵ-traced by some point $x \in X$ if $d(x_g, T_g(x)) < \epsilon$ for every $g \in G$. We say that an continuous action T has the shadowing property with respect to A if for every $\epsilon > 0$, there exists $\delta > 0$ such that each δ-pseudo orbit for T with respect to A is ϵ-traced by some point of X (See [1, 10]).

Recall the notion of shadowable points of a finitely generated group action on a compact metric space.

Definition 2.1 ([4]). A point x is shadowable with respect to A if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $\{x_g\}_{g\in G}$ of T with respect to A with $x_e = x$ can be ε-traced. We denote $Sh(T, A)$ by the set of shadowable points of T with respect to A.

It is clear that the definition of shadowable points of T does not depend on the choice of a compatible metric d on X. Furthermore, we can see that the concept of shadowable points of T does not depend on the choice of a symmetric finitely generating set A of G.

Definition 2.2 ([4]). We say that a point x is shadowable if x is shadowable with respect to A for a finitely generating set A of G. We denote $Sh(T)$ by the set of shadowable points of T.

We give some properties of $Sh(T)$ through the following standard definitions. Recently, Oprocha introduced the notion of the behavior of multidimensional time discrete dynamical systems containing the concepts of nonwandering set and chain recurrent set in [9]. The notion of definitions of finitely generated group action versions were introduced as follows:

Definition 2.3 ([4]). A point $x \in X$ is nonwandering of T with respect to A if for any neighborhood U of x and $g \in G$ with $l_A(g) > 0$ there exists $h \in G$ such that $l_A(h) > l_A(g)$ and $T_hU \cap U \neq \emptyset$. The set of all such nonwandering points is denoted by $\Omega(T, A)$.

Recall that let A be a finitely generating set of G. For any $k \in \mathbb{N}$, we put $B(k) = \{g \in G : l_A(g) \leq k\}$ where $l_A(g)$ is the word length metric on G induced by A.

Lemma 2.4. *A point $x \in \Omega(T, A)$ if and only if for any neighborhood U of x, there exist $h \in G\backslash\{e\}$ such that $T_h(U) \cap U \neq \emptyset$.*

Definition 2.5. Let $T \in Act(G,X)$ be an action with respect to a finitely generating set A of G. A point $x \in X$ is said to be chain recurrent of T with respect to A if for every $\delta > 0$ there exists a δ-pseudo orbit $\{x_g\}_{g\in G}$ of T with respect to A such that

(i) $x_e = x$,
(ii) if the equality $x_{g'} = x$ holds for some index $g' \in G$, then the set $\{h \in G : x_h = x,\ l_A(h) > g\}$ is infinite for all $g \in G$.

The set of all chain recurrent points with respect to A will be denoted by $CR(T,A)$. Every δ-pseudo orbit of T with respect to A satisfying condition (i) and (ii) is said to be a δ-chain for x.

The definitions of nonwandering points and chain recurrent points of T do not depend on the choice of a symmetric finitely generating set A of G.

Definition 2.6. Let $T \in Act(G,X)$ be an action with respect to a finitely generating set A of G. A set $E \subset X$ is called T-invariant set with respect to A if $T_{a^{-1}}(E) = E$ for all $a \in A$.

We consider the concept of weakly periodic points for T which is motivated by [2].

Definition 2.7. We say that $x \in X$ is periodic for T if $T_g x = x$ for some $l_A(g) > 0$. Moreover, a point $x \in X$ is called weakly periodic for T with respect to A if for every $\delta > 0$ there is a continuous action $S \in Act(G,X)$ with $d_A(T,S) < \delta$ such that x is periodic point of S. In other word, x satisfies the C^0-closing lemma with respect to d_A. Denoted $CL(T,A)$ by the set of weakly periodic points for T with respect to A and $Per(T)$ by the set of periodic points.

Lemma 2.8. *Let A and B be finitely symmetric generating set of G. Then $CL(T,A) = CL(T,B)$.*

Proof. Similarly to the proof of Lemma 2.8 in [4].

□

3. Continuous shadowable points

In this section, we introduced the notion of continuous shadowable points. Continuous shadowing was introduced and discussed in [6]. Let X^G be the compact metric space of all sequences $\xi = \{x_g : g \in G\}$ with elements $x_g \in X$, endowed with the product topology. Suppose that $G = \{g_i | i \in \mathbb{N}\}$ is countable, we define a metric D on X^G by

$$D((x_{g_i})_{i\in\mathbb{N}}, (y_{g_i})_{i\in\mathbb{N}}) = \sup_i \left\{ \frac{\bar{d}(x_{g_i}, y_{g_i})}{2^i} \right\}$$

for any $(x_{g_i})_{i\in\mathbb{N}}, (y_{g_i})_{i\in\mathbb{N}} \in X^G$ where $\bar{d}(x_{g_i}, y_{g_i}) = \min\{d(x_{g_i}, y_{g_i}), 1\}$. For $\delta > 0$, let $\Phi_T(\delta, A)$ denote the set of all δ-pseudo orbits of T with respect to A. An action $T \in Act(G, X)$ is said to be *continuous shadowing* with respect to A if for every $\epsilon > 0$, there exists $\delta > 0$ and a continuous map $r : \Phi_T(\delta, A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for any $\mathbf{x} = \{x_g\}_{g\in G} \in \Phi_T(\delta, A)$ and all $g \in G$. Fix $\varepsilon > 0$. For any $x \in Sh(T)$, we can choose δ_x of x for given ε. Before we introduce the notions of continuous shadowable points, first we define $\Phi_T(x, \delta_x, A)$ is the set of all δ_x-pseudo orbit of T with respect to A through x.

Definition 3.1. A point x is continuous shadowable of T with respect to A if for any $\epsilon > 0$, there exists $\delta_x > 0$ and a continuous map $r : \Phi_T(x, \delta_x, A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \Phi_T(x, \delta_x, A)$ and $g \in G$. We denote $CSh(T, A)$ by the set of continuous shadowable points of T with respect to A.

Lemma 3.2. *Let A and B be finitely symmetric generating sets of G. For any $T \in Act(G, X)$, if $x \in X$ is a continuous shadowable point of T with respect to A, then it is a continuous shadowable point of T with respect to B.*

Proof. Let $\varepsilon > 0$. Since $x \in CSh(T, A)$ there are $\delta_x > 0$ and a continuous map $r : \Phi_T(x, \delta_x, A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \Phi_T(x, \delta_x, A)$ with $x_e = x$ and $g \in G$. For each $x \in CSh(T, A)$, we claim that there exists a $\delta'_x > 0$ such that $\Phi_T(x, \delta'_x, B) \subset \Phi_T(x, \delta_x, A)$. Put $m = \max_{b\in B} l_B(a)$, where l_B is the word length metric on G induced by B. Choose $\delta_1 > 0$ such that $m\delta_1 < \delta_x$. Since X is compact and B is a finite set, there exists $0 < \delta'_x < \delta_1$ such that $d(T_h(x), T_h(y)) < \delta_1$ for $x, y \in X$ with $d(x, y) < \delta'_x$ and for $h \in G$ with $l_B(h) \leq m$. For any $a \in A$, we write a as $b_1 \cdots b_{l(a)}$ where $l(a) = l_B(a)$, $b_i \in B$ and $i = 1, \cdots, l(a)$. Then for any $\{x_g\}_{g\in G} \in \Phi_T(x, \delta'_x, B)$, we have

$$
\begin{aligned}
d(T_a(x_g), x_{ag}) &= d(T_{b_1\cdots b_{l(a)}}(x_g), x_{b_1\cdots b_{l(a)}g}) \\
&\leq d(T_{b_1\cdots b_{l(a)}}(x_g), T_{b_1\cdots b_{l(a)-1}}(x_{b_{l(a)}g})) \\
&\quad + d(T_{b_1\cdots b_{l(a)-1}}(x_{b_{l(a)}g}), T_{b_1 b_2\cdots b_{l(a)-2}}(x_{b_{l(a)-1}b_{l(a)}g})) \\
&\quad + \cdots + d(T_{b_1 b_2}(x_{b_3\cdots b_{l(a)}g}), T_{b_1} x_{b_2\cdots b_{l(a)}g}) \\
&\quad + d(T_{b_1} x_{b_2\cdots b_{l(a)}g}, x_{b_1\cdots b_{l(a)}g}) \\
&< (m-1)\delta_1 + \delta'_x < (m-1)\delta_1 + \delta_1 = m\delta_1 < \delta_x.
\end{aligned}
$$

This means that $\{x_g\}_{g\in G} \in \Phi_T(x, \delta_x, A)$. Let $\bar{r}$ be a restriction of r from $\Phi_T(x, \delta'_x, B)$ to X. Then the continuous map $\bar{r} : \Phi_T(x, \delta'_x, B) \to X$ satisfies $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \Phi_T(x, \delta'_x, B)$ and $g \in G$. Thus the point x is continuous shadowable of T with respect to B. $\square$

Definition 3.3. A point $x \in X$ is continuous shadowable if x is a continuous shadowable with respect to a finitely generating set A of G. We denote $CSh(T)$ by the set of continuous shadowable points of T.

Remark 3.4. Morales [7] showed that T has the shadowing property if and only if $Sh(T) = X$ where X is a compact metric space. If T has a continuous shadowing property then $CSh(T) = X$. But converse is not true. In fact, Yano [13] constructed an example of a homeomorphism f on unit circle S^1 which is shadowing but it is not topologically stable, and so it is not continuous shadowing by Theorem 2.5 in [6]. If $CSh(f) = S^1$ then f has the shadowing property but it not continuous shadowing. And this implies there is a point $x \in Sh(f) = S^1$ but $x \notin CSh(f)$. This means that $CSh(f) \neq Sh(f)$.

Definition 3.5. Let $T \in Act(G, X)$ be an action with respect to a finitely generating set A of G. A set $E \subset X$ is called a T-invariant set with respect to A if $T_{a^{-1}}(E) = E$ for all $a \in A$.

With these definitions we state the first result.

Theorem 3.6. *Let G be a finitely generated group. If $T \in Act(G, X)$ is a continuous action of G on compact metric space X, then $CSh(T)$ is an invariant subset of X.*

Proof. Let A be a finitely generating set of G. It suffices to prove that if $x \in CSh(T)$ then $T_a(x) \in CSh(T)$ for every $a \in A$. Fix $a \in A$ and $\varepsilon > 0$. Since T_a is uniformly continuous, so there is $0 < \varepsilon' < \varepsilon$ such that $d(T_a(y), T_a(z)) \leq \varepsilon$ whenever $y, z \in X$ satisfy $d(y, z) \leq \varepsilon'$ for all $a \in A$. For this ε', we get $\delta'_x > 0$ by the property of $x \in CSh(T)$. Since X is compact and A is finite, there is $0 < \delta_x < \frac{\delta'_x}{2}$ such that for any

$y, z \in X$, $d(y,z) < \delta_x$ implies $d(T_{a'}(y), T_{a'}(z)) < \frac{\delta'_x}{2}$ for any $a' \in A$ and $a' \neq a$. Let $\mathbf{x} = \{x_g\}_{g\in G}$ be a δ_x-pseudo orbit of T with respect to A through $T_a(x)$. We define a new pseudo orbit $\mathbf{y} = \{y_g\}_{g\in G}$ by

$$y_g = \begin{cases} T_{a^{-1}}(x_e), & \textit{if } \ g = e, \\ x_{ga^{-1}}, & \textit{if } \ g \neq e. \end{cases}$$

Then we can check $\{y_g\}_{g\in G}$ is a δ'_x-pseudo orbit of T with respect to A through x. Indeed,
if $g = e$ and $a' = a$, then we have

$$\begin{aligned} d(T_{a'}(y_g), y_{a'g}) &= d(T_{a'}(y_e), y_{a'}) = d(T_{a'}(T_{a^{-1}}(x_e)), y_{a'}) \\ &= d(x_e, y_a) = d(x_e, x_e) = 0 < \delta'_x. \end{aligned}$$

If $g = e$ and $a' \neq a$, then we get

$$\begin{aligned} d(T_{a'}(y_g), y_{a'g}) &= d(T_{a'}(y_e), y_{a'}) = d(T_{a'}(T_{a^{-1}}(x_e)), y_{a'}) \\ &\leq d(T_{a'}(T_{a^{-1}}(x_e)), T_{a'}(x_{a^{-1}})) + d(T_{a'}(x_{a^{-1}}), y_{a'}) \\ &< \frac{\delta'_x}{2} + d(T_{a'}(x_{a^{-1}}), x_{a'a^{-1}}) < \frac{\delta'_x}{2} + \delta_x < \delta'_x. \end{aligned}$$

If $g \neq e$, then we obtain

$$d(T_{a'}(y_g), y_{a'g}) = d(T_{a'}(x_{ga^{-1}}), x_{a'ga^{-1}}) < \delta'.$$

Since $x \in CSh(T)$, there is a continuous map $r : \Phi_T(x, \delta'_x, A) \to X$ such that $d(T_g(r(\mathbf{y})), y_g) < \varepsilon'$, for all $\mathbf{y} = \{y_g\}_{g\in G} \in \Phi_T(x, \delta'_x, A)$ and $g \in G$. Thus $d(T_g(r(\mathbf{y})), y_g) < \varepsilon'$ for all $g \in G$. Then for $g = e$, we deduce

$$d(T_e(r(\mathbf{y})), y_e) = d(r(\mathbf{y}), T_{a^{-1}}x_e) = d(T_{a^{-1}}(T_a(r(\mathbf{y}))), T_{a^{-1}}(x_e)) < \varepsilon'.$$

So, we have $d(T_a(r(\mathbf{y})), x_e) = d(T_e(T_a(r(\mathbf{y}))), x_e) < \varepsilon$. For $g \neq e$, we get

$$\begin{aligned} d(T_g(r(\mathbf{y})), y_g) &= d(T_g(r(\mathbf{y})), x_{ga^{-1}}) = d(T_{g'a}(r(\mathbf{y})), x_{g'}) \\ &= d(T_{g'}(T_a(r(\mathbf{y}))), x_{g'}) < \varepsilon' < \varepsilon. \end{aligned}$$

We can define a continuous map $f : \Phi_T(T_a x, \delta_x, A) \to \Phi_T(x, \delta'_x, A)$ by $f(\mathbf{x}) = \mathbf{y}$ as previously defined. In fact, for any $\varepsilon' > 0$, we can choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon'$. Choose $\delta = \frac{1}{2^{n+1}}$. Let $\mathbf{x} = \{x_g\}_{g\in G}, \mathbf{y} = \{y_g\}_{g\in G} \in \Phi_T(T_a(x), \delta_x, A)$. If $D(\mathbf{x}, \mathbf{y}) < \delta$ then $x_g = y_g$ for all $g \in B(n+1)$. This implies $x_{ga^{-1}} = y_{ga^{-1}}$ for all $g \in B(n)$. Thus $D(f((x)), f((y))) = \frac{1}{2^n} < \varepsilon'$. So f is continuous. Then the map $\alpha = T_a \circ r \circ f : \Phi_T(T_a(x), \delta_x, A) \to X$ is continuous and satisfies $d(T_g(\alpha(\mathbf{x})), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \Phi_T(T_a(x), \delta_x, A)$ and $g \in G$. Thus $T_a(x) \in CSh(T)$. Hence $CSh(T)$ is invariant.

□

Morales [7] showed that the set of shadowable points is not compact in general. But, Kawaguchi [3] proved that the set of shadowable points is a Borel set. Likewise we will show that the set of continuous shadowable points is a Borel set.

THEOREM 3.7. *If $T \in Act(G, X)$ then $CSh(T)$ is a Borel set.*

Proof. Let A be a finitely generating set of G. For any $\delta > 0$ and $c > 0$, let $CSh_{\delta,c}(T, A)$ be the set of points $z \in X$ such that there is a continuous map $r : \Phi_T(z, \delta, A) \to X$ satisfying $d(T_g(r(\mathbf{x})), x_g) < c$ for all $g \in G$, $\mathbf{x} \in \Phi_T(z, \delta, A)$. Then it is easy to see that

$$CSh_c(T, A) = \bigcup_{m\in\mathbb{N}} CSh_{\frac{1}{m},c}(T, A) \quad \text{and} \quad CSh(T, A) = \bigcap_{n\in\mathbb{N}} CSh_{\frac{1}{n}}(T, A).$$

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of points in $CSh_{\delta,c}(T, A)$ such that $\lim_{n\to\infty} z_n = z$ for some $z \in X$.

By Lemma 5.1 in [3], we know that $Sh_{\delta,c}(T, A)$ is a closed set. So $z \in Sh_{\delta,c}(T, A)$. Given a δ-pseudo orbit $\mathbf{x} = \{x_g\}_{g\in G}$ with $d(T_a(x_g^{(n)}), x_{ag}^{(n)}) < \delta$ for all $a \in A$ and $g \in G$ and $x_e = x$, we define a sequence of pseudo orbits $\mathbf{x}^{(n)} = \{x_g^{(n)}\}_{g\in G}$ ($n \in \mathbb{N}$) by $x_e^{(n)} = z_n$, $\mathbf{x}_g^{(n)} = x_g$ for $g \neq e$. Then for sufficiently large $n \in \mathbb{N}$, we have $d(T_a(x_g^{(n)}), x_{ag}^{(n)}) < \delta$ for all $a \in A$, $g \in G$. And also $z_n \in CSh_{\delta,c}(T, A)$. Hence there exists a continuous map $r_n : \Phi_T(z_n, \delta, A) \to X$ such that every $\mathbf{x}^{(n)} = \{x_g^{(n)}\}_{g\in G} \in \Phi_T(z_n, \delta, A)$ is c-traced by $r_n(x^{(n)})$. Since X is compact, this sequence $r_n(\mathbf{x}^{(n)})$ converges to some point in X. Then we can easily to show that $\mathbf{x}$ is c-traced by $\alpha = \lim_{n\to\infty} r_n(\mathbf{x}^{(n)})$. Define a map $r : \Phi_T(z, \delta, A) \to X$ by $r(\mathbf{x}) = \alpha$ for all $\mathbf{x} \in \Phi_T(z, \delta, A)$. We will show that r is continuous. For any $\varepsilon > 0$ and $\mathbf{x} \in \Phi_T(z, \delta, A)$, we can choose $m \in \mathbb{N}$ such that $d(r_m(\mathbf{x}^{(m)}), r(\mathbf{x})) < \frac{\varepsilon}{3}$. For given $\frac{\varepsilon}{3} > 0$, since r_m is continuous at $\mathbf{x}^{(m)}$, there exists $\delta > 0$ such that $d(\mathbf{y}^{(m)}, \mathbf{x}^{(m)}) < \delta$ implies $d(r_m(\mathbf{y}^{(m)}), r_m(\mathbf{x}^{(m)})) < \frac{\varepsilon}{3}$ for $\mathbf{y}^{(m)} \in \Phi_T(z_m, \delta, A)$. Choose $N \in \mathbb{N}$ with $\frac{1}{2^N} < \delta$. Then $y_g^{(m)} = x_g^{(m)}$ for $g \in B(N) = \{h \in G : l_A(h) \leq N\}$. Let $\mathbf{y} = \{y_g\}_{g\in G}$ by $y_e = z$ and $y_g = y_g^{(m)}$ for $g \neq e$. Then $D(\mathbf{x}, \mathbf{y}) < \delta$ implies

$$\begin{aligned} d(r(\mathbf{x}), r(\mathbf{y})) \leq d(r(\mathbf{x}), r_m(\mathbf{x}^{(m)})) + d(r_m(\mathbf{x}^{(m)}), , r_m(\mathbf{y}^{(m)})) \\ + d(r_m(\mathbf{y}^{(m)}), r(\mathbf{y})) < \varepsilon. \end{aligned}$$

Thus r is a continuous map and $z \in CSh_{\delta,c}(T, A)$. So $CSh(T)$ is a Borel set. $\square$

4. Topologically stable points

Recently, Chung and Lee [1] introduced the notion of topological stability of a finitely generated group action on a compact metric space. We say that an action $T \in Act(G, X)$ is *topologically stable* with respect to a finitely generating set A of G if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$ then there exists a continuous map $f : X \to X$ with $T_g f = f S_g$ for every $g \in G$ and $d(f, Id_X) \leq \varepsilon$. An action T is said to be *topologically stable* if it is topological stable with respect to A for some finitely generating set A of G.

If T and S are two continuous actions of G on X with $d_A(T, S) < \delta$, then the S-orbit $\{S_g(x)\}_{g\in G}$ of $x \in X$ is nearly a T-orbit in the sense that $d(T_a(S_g(x)), S_{ag}(x)) < \delta$ for all $a \in A$ and $g \in G$. This observation motivates the following definition. Our definition of topologically stable points of a finitely generated group action also will be based on the following simple remark similarly [5]. Let $O_T(x) = \{T_g(x) : g \in G\}$ the orbit of $x \in X$ under T and $\overline{O_T(x)}$ is the orbit closure. Let $i_{\overline{O_T(x)}}$ be the inclusion $\overline{O_T(x)} \hookrightarrow X$ of a subset $\overline{O_T(x)} \subset X$.

Remark 4.1. A necessary condition for T to be topologically stable with respect to A is that every $x \in X$ satisfies the following property:

(P) For every $\varepsilon > 0$ there is $\delta_x > 0$ such that for every continuous action $S \in Act(G, X)$ satisfying $d_A(T, S) \leq \delta_x$ there is a continuous map $f : \overline{O_S(x)} \to X$ such that $d_{C^0}(f, i_{\overline{O_S(x)}}) \leq \varepsilon$ and $T_g \circ f = f \circ S_g$ for all $g \in G$.

Proof. Fix $x \in X$ and take $\delta_x = \delta$ where $\delta > 0$ is given by the topological stability of T with respect to A. If $S \in Act(G, X)$ with $d_A(T, S) \leq \delta_x$ then there is $k : X \to X$ continuous such that $d_{C^0}(k, Id_X)$ and $T_g \circ k = k \circ S_g$ for all $g \in G$. By taking f as the restriction $k|_{\overline{O_S(x)}}$ of k to $\overline{O_S(x)}$ we obtain (P). $\square$

This remark motivates the use of (P) as definition topologically stable point. This is the content of the main definition of this work.

Definition 4.2. We say that $x \in X$ is a topologically stable point of T with respect to A on a compact metric space X if for every $\varepsilon > 0$ there is $\delta_x > 0$ such that for every $S \in Act(G,X)$ satisfying $d_A(T,S) \leq \delta_x$ there is a continuous map $f : \overline{O_S(x)} \to X$ such that $d_{C^0}(f, i_{\overline{O_S(x)}}) \leq \varepsilon$ and $T_g \circ f = f \circ S_g$ for all $g \in G$. We denote $\mathcal{T}(T,A)$ by the set of topologically stable points of T with respect to A.

Lemma 4.3. *Let A and B be finitely symmetric generating sets of G. Then $\mathcal{T}(T,A) = \mathcal{T}(T,B)$.*

Proof. Suppose $x \in \mathcal{T}(T,A)$. Then for any $\epsilon > 0$ there exists $\delta'_x > 0$ such that if S is another continuous action of G on X with $d_A(T,S) < \delta'_x$ then there exists a continuous map $f : \overline{O_S(x)} \to X$ with $T_g f = f S_g$, for all $g \in G$ and $d_{C^0}(f, i_{\overline{O_S(x)}}) \leq \varepsilon$. It suffices to show that there exists $\delta > 0$ such that for any $S \in Act(G,X)$, if $d_B(T,A) < \delta$ then $d_A(T,S) < \delta'$. Put $m = \max_{a \in A} l_B(a)$. Choose $\delta_1 > 0$ such that $m\delta_1 < \delta'_x$. Since X is compact, T_g is uniformly continuous. Given δ', we can choose δ_x such that

$$d(x,y) < \delta_x \text{ implies } d(T_g(x), T_g(y)) < \delta_1 \text{ for all } g \in B(m)$$

where $B(m) = \{g \in G : l_B(g) \leq m\}$. For any $a \in A$, we write a as $b_1 \cdots b_n$, where $n = l_B(a) \leq m$, $b_i \in B$, $i = 1, \cdots, n$. Then we have

$$\begin{aligned} d(T_a(x), S_a(x)) &= d(T_{b_1 \cdots b_n}(x), S_{b_1 \cdots b_n}(x)) \\ &\leq d(T_{b_1 \cdots b_{n-1}}(T_{b_n}(x)), T_{b_1 \cdots b_{n-1}}(S_{b_n}(x))) \\ &\quad + d(T_{b_1 \cdots b_{n-2}} T_{b_{n-1}}(S_{b_n}(x)), T_{b_1 \cdots b_{n-2}} S_{b_{n-1}}(S_{b_n}(x))) \\ &\quad + \cdots + d(T_{b_1} T_{b_2} S_{b_3 \cdots b_{n-1} b_n}(x), T_{b_1} S_{b_2} S_{b_3 \cdots b_{n-1} b_n}(x)) \\ &\quad + d(T_{b_1} S_{b_2 \cdots b_{n-1} b_n}(x), S_{b_1 \cdots b_n}(x)) \\ &< m\delta_1 < \delta'_x \end{aligned}$$

This means that $d_A(T,S) < \delta'_x$, and so completes the proof. □

Definition 4.4. We says that $x \in X$ is a topologically stable point of $T \in Act(G,X)$ if it is a topologically stable point of T with respect to a finitely generating set A of G.

By Remark 4.1 if T is topologically stable then every point is also topologically stable. We do not know if the converse holds. Morales [5] introduced an example about topological stable points of homeomorphisms. We can adapt the following examples of group action versions.

Example 4.5. *Let $Id_{G \times X} : G \times X \to X$ be the identity group action. For every metric space X one has that $\mathcal{T}(Id_{G \times X}, A)$ consists of those points $x \in X$ for which for every $\epsilon > 0$ there is $\delta > 0$ such that*

$d(x, S_g x) \leq \delta$ for every $g \in G$ for every $S \in Act(G, X)$ with $d_A(T, S) \leq \delta$. This implies that $\mathcal{T}(Id_{G\times X}, A)$ contains the isolated points of X. It also implies that $\mathcal{T}(Id_{G\times X}, A) = \emptyset$ when X is a manifold.

EXAMPLE 4.6. *Suppose that X is a metric space and $x_0 \in X$ is a fixed point for every $T \in Act(G, X)$. Then $x_0 \in \mathcal{T}(T, A)$ for every $T \in Act(G, X)$. By applying this remark to the union X of two circles with a common point x_0 we have that $\mathcal{T}(Id_{G\times X}, A) = \{x_0\}$. Hence there are continuous actions on uncountable compact metric space exhibiting a unique topologically stable point.*

EXAMPLE 4.7. *By the previous example the set of topologically stable points need not be open. Such a set need not be closed too by the following counter example. Define $X = [0, 1] \cup \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ equipped with the Euclidean metric. Clearly $\mathcal{T}(Id_{G\times X}) = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ is not a closed subset of X.*

Now we present some properties of the set of topologically stable points. Previously we establish some short definitions. Let $T \in Act(G, X)$ be a continuous action of a finitely generated group G on a compact metric space X. We say that T is minimal if $O_T(x)$ is dense in X for every $x \in X$ and denote $Int(Per(T))$ by the set of the interior points of periodic points for T.

THEOREM 4.8. *The following properties hold for every action $T \in Act(G, X)$ of every finitely generated group G and a compact metric space X.*

1. *$\mathcal{T}(T)$ is an invariant set of T.*
2. *$CL(T) \cap \mathcal{T}(T) \subset \overline{Per(T)}$. In particular, if X is infinite and T is minimal, then $CL(T) \cap \mathcal{T}(T) = \emptyset$.*
3. *If X is infinite and T can be C^0-approximated with respect to d_A by minimal actions, then $Int(Per(T)) \cap \mathcal{T}(T) = \emptyset$.*

LEMMA 4.9. *The set of topologically stable points of an action $T \in Act(G, X)$ is invariant.*

Proof. Let A be a finitely generating set of G. Fix $x \in \mathcal{T}(T)$ and $a \in A$. It suffices to prove that $T_a(x) \in \mathcal{T}(T)$. For any $\epsilon > 0$, since T_a is uniformly continuous there is $\varepsilon' > 0$ such that $d(T_a(x), T_a(y)) < \varepsilon$ whenever $y, z \in X$ satisfy $d(x, y) < \varepsilon'$. For this ε', we let $\delta'_x > 0$ be given by the topological stability of x with respect to A. Once again using uniformly continuity of T, we can choose $0 < \delta_x < \frac{\delta'_x}{3}$ such that $d(x, y) < \delta_x$ implies $d(T_{a'}^{-1}(x), T_{a'}^{-1}(y)) < \frac{\delta'_x}{3}$ for all $a' \in A$. Now take an action $S \in Act(G, X)$ with $d_A(T, S) < \delta_x$. Then the choice of δ_x implies $d(T_{a'}(T_a(x)), S_{a'}(T_a(x))) < \delta_x$ for all $a' \in A$ and $x \in X$. Define a continuous action $\tilde{S} : G \times X \to X$ by

$$\tilde{S}_g(x) = \begin{cases} T_a^{-1} \circ S_g \circ T_a(x) & g = e, \\ T_a^{-1} \circ S_{aga^{-1}} \circ T_a(x) & g \neq e. \end{cases}$$

For any $a' \in A$ and $x \in X$,

$$\begin{aligned} d(\tilde{S}_{a'}(x), T_{a'}(x)) &= d(T_a^{-1} S_{aa'a^{-1}} T_a(x), T_{a'}(x)) \\ &= d(T_a^{-1} S_{aa'a^{-1}} T_a(x), T_{a^{-1}} T_{aa'a^{-1}} T_a(x)) \\ &= d(T_a^{-1} S_{aa'a^{-1}} T_a(x), T_a^{-1} T_a S_{a'a^{-1}} T_a(x)) \\ &\quad + d(T_a^{-1} T_a S_{a'a^{-1}} T_a(x), T_a^{-1} T_a T_{a'} S_{a^{-1}} T_a(x)) \\ &\quad + d(T_{a'} S_{a^{-1}} T_a(x), T_{a'} T_{a^{-1}} T_a(x)) \\ &< \frac{\delta'_x}{3} + \delta_x + \frac{\delta'_x}{3} < \delta'_x. \end{aligned}$$

So $d_A(T, \tilde{S}) < \delta'_x$. By definition of the topological stability of x, there is $h' : \overline{O_{\tilde{S}}(x)} \to X$ satisfying $d_{C^0}(h', i_{O_{\tilde{S}}(x)}) < \varepsilon'$ and $T_g \circ h' = h' \circ S_g$ for all $g \in G$. Define the continuous map $h = T_a \circ h' \circ T_a^{-1} : \overline{O_S(T_a(x))} \to X$. For any $y \in \overline{O_S(T_a(x))}$, since $d(h'(T_a^{-1}(y)), T_a^{-1} y) < \varepsilon'$ then $d(h(y), y) = d(T_a \circ h' \circ T_a^{-1}(y), y) < \varepsilon$. Moreover,

$$\begin{aligned} T_g \circ h &= T_g \circ T_a \circ h' \circ T_a^{-1} = T_a \circ T_{a^{-1}ga} \circ h' \circ T_a^{-1} \\ &= T_a \circ h' \circ \tilde{S}_{a^{-1}ga} \circ T_a^{-1} = T_a \circ h' \circ T_a^{-1} \circ S_g \circ T_a = h \circ S_g \end{aligned}$$

yielding $T_g \circ h = h \circ S_g$ for all $g \in G$. Thus $T_a(x) \in \mathcal{T}(T)$, so $\mathcal{T}(T)$ is an invariant set of T. □

LEMMA 4.10. *For every continuous action $T \in Act(G, X)$ of a compact metric space X one has $CL(T) \cap \mathcal{T}(T) \subseteq \overline{Per(T)}$. In particular, if X is infinite and T is minimal then $CL(T) \cap \mathcal{T}(T) = \emptyset$.*

Proof. Let A be a finitely generating set of G. Fix $x \in CL(T, A) \cap \mathcal{T}(T, A)$ and $\varepsilon > 0$. Take δ_x from the topological stability of x for this ε. Since $x \in CL(T, A)$, there is a $S \in Act(G, X)$ with $d_A(T, S) \leq \delta_x$ such that $x \in Per(S)$. Since $d_A(T, S) \leq \delta_x$, there is a continuous map $f : \overline{O_S(x)} \to X$ such that $d(f, i_{\overline{O_S(x)}}) \leq \varepsilon$ and $T_g \circ f = f \circ S_g$ for all $g \in G$. As $x \in Per(S)$, we have $S_g x = x$ for some $g \in G \setminus \{e\}$ and so $T_g(f(x)) = f(S_g(x)) = f(x)$ yielding $f(x) \in Per(T)$. Since $d(x, f(x)) \leq d_{C^0}(f, i_{\overline{O_S(x)}})$, we get a periodic point of T within ε from x. As ϵ is arbitrary, $x \in \overline{Per(T)}$ proving the first part of the lemma. For the second part if X is infinite and T is minimal we have $Per(T) = \emptyset$. So $\overline{Per(T)} = \emptyset$, thus $CL(T, A) \cap \mathcal{T}(T, A) = \emptyset$ by the first part. □

LEMMA 4.11. *Let $T \in Act(G, X)$ of an infinite metric space X. If T can be C^0-approximated with respect to d_A by minimal actions then $Int(Per(T)) \cap \mathcal{T}(T) = \emptyset$.*

Proof. Otherwise, there is $x \in Int(Per(T)) \cap \mathcal{T}(T)$. Fix $\varepsilon > 0$ such that $x \in Per(T)$ for every $y \in X$ with $d(x, y) \leq \varepsilon$. Now take $\delta_x > 0$ for this ε from the topological stability of x. By hypothesis there is a minimal action $S \in Act(G, X)$ with $d_A(T, S) \leq \delta_x$. Hence there is a continuous map $f : \overline{O_S(x)} \to X$ such that $d_{C^0}(h, i_{\overline{O_S(x)}}) \leq \epsilon$ and $T_g \circ f = f \circ S_g$. Again the latter identity implies $O_T(f(x)) = f(O_S(x))$. As S is minimal, $O_S(x)$ is dense in X and so $f(O_S(x))$ is dense in X by the continuity of f. It follows that $O_T(f(x))$ is dense in X. However $d(f(x), x) \leq \varepsilon$ so $f(x) \in Per(T)$ by the choice of ε thus $O_T(f(x))$ is a finite set. Hence X has a finite dense subset which contradicts that X is infinite. This ends the proof. □

Proof of Theorem 4.8. The proof follows from the previous results. □

5. The proof of main theorem

We recall that an action $T \in Act(G, X)$ is expansive if there exists a constant $\eta > 0$ called an expansive constant of T such that for every $x \neq y$, one has $\sup_{g \in G} d(T_g x, T_g y) > c$. With these definitions we have the following result.

THEOREM 5.1. *Let $T \in Act(G, X)$ be an expansive action of a compact metric space X. Then $CSh(T) = Sh(T) \subset \mathcal{T}(T)$*

We decompose the proof of Theorem 5.1 into the following lemmas.

LEMMA 5.2. *Let $T \in Act(G, X)$ be an expansive action of a compact metric space X. If $x \in Sh(T)$ then $x \in CSh(T)$.*

Proof. Let $\eta > 0$ be an expansive constant of T and A be a finitely generating set of G. For $0 < \varepsilon < \frac{\eta}{3}$ and $x \in Sh(T)$ there exists $\delta_x > 0$ correspond to ε and x by the shadowing of T with respect to A. Then for any $x \in Sh(T)$ and δ_x-pseudo orbit $\{x_g\}_{g \in G}$ of T with respect to A through x there exists a unique $y \in X$ satisfying $d(T_g(y), x_g) < \varepsilon$ for all $g \in G$. Hence we can define a map $r : \mathcal{P}(T, A) \to X$ by $r(\{x_g\}_{g \in G})$

is the ε-shadowing point of $\{x_g\}_{g\in G} \in \mathcal{P}_x(T,\delta_x,A)$. We claim that r is continuous. Since T is expansive, for any $\varepsilon' > 0$ there exists $B(M)$ is finite subset of G such that $\sup_{g\in B(M)} d(T_g(x),T_g(y)) \le \eta$ then $d(x,y) < \varepsilon'$ where $B(M) = \{g \in G \mid l_A(g) \le M\}$ by Lemma 2.10 in [1]. Pick $\delta' > 0$ with $2^M\delta' < \frac{\eta}{3}$. Let $\mathbf{x} = \{x_g\}_{g\in G}$, $\mathbf{x}' = \{x'_g\}_{g\in G} \in \mathcal{P}_\alpha(T,\delta,A)$ be given two δ-pseudo orbits of T with respect to A, and let $r(\mathbf{x}) = y$ and $r(\mathbf{x}') = y'$. If $D(\{x_g\}_{g\in G},\{x'_g\}_{g\in G}) < \delta'$ then $d(x_g,x'_g) < \frac{\eta}{3}$ for all $g \in B(M)$. We have

$$d(T_g(y),T_g(y')) \le d(T_g(y),x_g) + d(x_g,x'_g) + d(x'_g,T_g(y')) < \varepsilon + \tfrac{\eta}{3} + \varepsilon < \eta$$

for all $g \in B(M)$. So we have $d(y,y') < \varepsilon'$. Therefore r is continuous. □

Remark 5.3. Walters' shadowing lemma asserts that every topologically stable homeomorphism of a compact manifold of dim ≥ 2 has a shadowing property and Koo *et al.* [5] asserts that pointwise version of Walters' shadowing lemma.

Walters' stability Theorem [12] asserts that every expansive homeomorphism with the shadowing property of a compact metric space is topologically stable. Recently Chung and Lee [1] represent a group action version of the Walters' stability theorem. We also have a pointwise version of this result.

Lemma 5.4. *Every shadowable point of an expansive continuous action of a compact metric space is topologically stable.*

Proof. Let $T \in Act(G,X)$ is expansive and let A be a finitely generating set of G. Fix an expansive constant η of T and $0 < \varepsilon' < \frac{1}{4}\min\{\eta,\varepsilon\}$. For this ε', we fix $\delta > 0$ from $x \in Sh(T)$. Let S be a continuous action G on X with $d_A(T,S) \le \delta$. It follows that the S-orbit $\{S_g(x)\}_{g\in G}$ of x is a δ-pseudo orbit of T with respect to A and so there is $y \in X$ such that $d(T_g(y),S_g(x)) \le \varepsilon'$ for every $g \in G$. Now we define the map $f : O_S(x) \to X$ by $h(z) = T_g(y)$ whenever $z = S_g(x)$ for some $g \in G$. To prove that this map is well-defined we need to show that $T_{g_1}(y) = T_{g_2}(y)$ whenever $g_1,g_2 \in G$ satisfy $S_{g_1}(x) = S_{g_2}(x)$. Indeed if $S_{g_1}(x) = S_{g_2}(x)$ for some $g_1,g_2 \in G$, we have $S_{gg_1}(x) = S_{gg_2}(x)$ for every $g \in G$ and so

$$\begin{aligned} d(T_g(T_{g_1}(y)),T_g(T_{g_2}(y))) &\le d(T_{gg_1}(y),S_{gg_1}x) + d(S_{gg_1}(x),S_{gg_2}(x)) \\ &\quad + d(S_{gg_2}(x),T_{gg_2}(y)) \\ &= d(T_{gg_1}(y),S_{gg_1}(x)) + d(S_{gg_2}(x),T_{gg_2}(y)) \\ &\le 2\epsilon' < \eta \end{aligned}$$

for every $g \in G$. Since η is an expansive constant, we obtain $T_{g_1}(y) = T_{g_2}(y)$ proving the assertion. It follows that the map f defined above is well-defined. Moreover if $z = S_g(x)$ for some $g \in G$ one has

$$T_a(f(z)) = T_a(T_a(y)) = T_{ag}(y) = f(S_{ag}(x)) = f(S_a(S_g(x))) = f(S_a(z))$$

for all $a \in A$. This proves $T_a \circ f = f \circ S_a$ for all $a \in A$. Again, if $z = S_g(x)$ for some $g \in G$,

$$d(f(z), z) = d(f(S_g(x)), S_g(x)) = d(T_g(y), S_g(x)) \leq \varepsilon' < \varepsilon$$

thus proving $d_{C^0}(f, i_{O_S(x)}) \leq \varepsilon$.

Let us prove that f is uniformly continuous. Fix $\varepsilon_1 > 0$. Since η is an expansivity constant and X is a compact metric space, we have from Lemma 2.10 in [1] that there exists a non-empty finite subset F of G such that whenever $\sup_{g \in F} d(T_g(x), T_g(y)) \leq \eta$ one has $d(x, y) < \varepsilon_1$. Choose $\delta_1 > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta_1$ one has $d(S_g(x), S_g(y)) < \frac{\eta}{3}$ for every $g \in F$. Then for any $x, y \in X$ with $d(x, y) < \delta_1$ and $g \in F$, we get

$$\begin{aligned} d(T_g f(x), T_g f(y)) &= d(f(S_g(x)), f(S_g(y))) \\ &\leq d(f(S_g(x)), S_g(x)) + d(S_g(x), S_g(y)) \\ &\quad + d(S_g(y), f(S_g(y))) \\ &< \varepsilon + \frac{\eta}{3} + \varepsilon < \eta. \end{aligned}$$

Thus $d(f(x), f(y)) < \varepsilon_1$ for $x, y \in X$ with $d(x, y) < \delta_1$. This means that f is uniformly continuous. Then we can extend f continuously to the orbit closure $\overline{O_S(x)}$ to obtain a continuous map still denoted by $\overline{f} : \overline{O_S(x)} \to X$ satisfying $d_{C^0}(\overline{f}, i_{\overline{O_S(x)}}) \leq \varepsilon$ and $T_g \circ \overline{f} = \overline{f} \circ S_g$ for all $g \in G$. It follows that $x \in \mathcal{T}(T)$ and so $Sh(T) \subseteq \mathcal{T}(T)$. This ends the proof. □

Proof of Theorem 5.1. If an action T be an expansive action on compact metric space, $CSh(T) = Sh(T)$ and $Sh(T) \subset \mathcal{T}(T)$ by Lemma 5.2 and Lemma 5.4. □

References

[1] N.P. Chung and K. Lee, *Topological stability and pseudo-orbit tracing property of group actions*, Proc. Amer. Math. Soc., **146** (2018), 1047–1057.

[2] S. Choi, C. Chu, and K. Lee, *Recurrence in persistent dynamical systems*, Bull. Austral. Math. Soc., **43** (1991), 509–517.

[3] N. Kawaguchi, *Quantitative shadowable points*, Dyn. Syst., **32** (2017), 504–518.

[4] S. Kim and K. Lee, *Shadowable points for finitely generated group actions*, J. Chungcheong Math. Soc., **31** (2018), 411–420.
[5] N. Koo, K. Lee and C.A. Morales, *Pointwise topological stability*, Proc. Edinburg Math. Soc., **61** (2018), 1179–1191.
[6] K. Lee, *Continuous inverse shadowing and hyperbolicity*, Bull. Austral. Math. Soc., **67** (2003), 15–26.
[7] C.A. Morales, *Shadowable points*, Dyn. Syst., **31** (2016), 347–356.
[8] A.V. Osipov and S.B. Tikhomirov, *Shadowing for actions of some finitely generated groups*, Dyn Syst., **29** (2014), 337–351.
[9] P. Oprocha, *Chain recurrence in multidimensional time discrete dynamical systems*, Discrete Contin. Dyn. Syst., **20** (2008), 1039–1056.
[10] S.Y. Pilyugin, *Inverse shadowing in group action*, Dyn. Syst., **32** (2017), 198–210.
[11] S.Y. Pilyugin and S.B. Tikhomirov, *Shadowing in actions of some abelian groups*, Fund. Math., **179** (2003), 83–96.
[12] P. Walters, *On the pseudo-orbit tracing property and its relationship to stability*, Lecture Notes in Math. **668**, Springer, Berlin. 1978, 231–244.
[13] K. Yano, *Topologically stable homeomorphisms of the circle*, Nagoya Math., **79** (1980), 145–149.

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